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Need for Knowledge of Trigonometry and Characteristics of Its Teaching

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Abstract: In the present work, in the first place, the need for knowledge of trigonometry is argued, both in Mathematics itself and in practical applications; but despite the usefulness and necessity of trigonometry, it can be seen in the specialized bibliography that it is usual for students to present difficulties with the aforementioned contents. For this reason, the work presented here is aimed at determining aspects of this branch of Mathematics that influence the difficulties that students present in their learning; Reasons for these difficulties are specified and precise didactically based guidelines are provided in order to avoid or at least mitigate them. Therefore, the objective of this work is to provide a set of didactic guidelines, with their theoretical foundations, that help teachers in the direction of the teaching-learning process of trigonometry. Obviously, as reflected in the specialized bibliography, the student's activity is essential for their learning and cognitive development. The theoretical foundations that serve as a basis in a specific way in the proposed proposal are also specified, such as representation register changes, theoretical generalization, and process-to-object transit. The methods used are based on bibliographic review and participant research with a pedagogical experiment to check the results obtained.

Keywords: Trigonometry, Didactic Teaching, Semiotic Registers, Generalization, ICT

1. Introduction

The development of science and technology demands an increase in the quality of human resources. One way to increase the human resources required is through Mathematics education, given the important role of this science in daily life, through learning Mathematics it is expected that students consolidate the ability to think logically, critically, systematically, effectively and efficiently, which fosters their ability to solve problems.

Trigonometry is a fundamental content in the systemic structure of Mathematics, due to the use of trigonometric contents both in applications in problems external to Mathematics or given its use as a tool for the development of Mathematics itself, which highlights aspects that characterize the ontology of Mathematics, the fact of being a means and an object in itself, as well as the systemic structure of this science Báez & Blanco [1]. Despite the importance of trigonometry in Mathematics itself and in science in general, research shows that there are difficulties in these topics for students and even for teachers, Serpe & Frassiaa [14], Several studies have common conclusions that the traditional methods in teaching trigonometry are inadequate to introduce the students into a trigonometric function's concept Jerito & Hermita [7].

Undoubtedly, trigonometry is a specific branch of Mathematics, but from the point of view of its teaching, the treatment of trigonometric ratios for the study of metric relations in triangles is not the same as its generalization as functions of real variables, with their own characteristics that distinguish them from other functions that are studied at the intermediate level, such as the logarithmic and exponential functions, since the trigonometric functions are not bijective and are also periodic unlike those mentioned above. According to studies carried out, it is suggested that the traditional teaching of trigonometry, by not relating the trigonometry of triangles with the trigonometry of trigonometric functions, is one of the reasons that cause difficulties in the teaching-learning process of these contents. Serpe & Frassia [12].

Given the high level of intra and extra mathematical applications of trigonometry, its study should be deepened, since Eratosthenes Greek Mathematics used trigonometry to measure the diameter of the earth, in civil engineering it is required to measure the angle of cant in the curves, through which the hyperbolic functions are defined, which are used to study the hanging rope problem, among others, appear in the Fourier series, which are necessary for the generalized solution of the partial differential equations for the study of vibration problems and heat conduction, are necessary in the resolution of integrals of rational functions and other integrals with radicals in the integrand, in their relations with the exponential functions of base e, giving rise to Euler's equations, from where the equality: $e^{\pi i} = -1$ is derived, which shows without a doubt the systemic structure of Mathematics.

Speaking specifically of the teaching of trigonometry, the interrelationships that are manifested between the trigonometric functions should be taken advantage of and give rise to necessary identities in different mathematical applications, these interrelationships allow students to be shown how Mathematics is built with mathematical tools, avoiding a rote teaching so that students can work with the resources and knowledge they will need in the long term Doherty, Bellestier, & Rhodes, [4].

2. Theoretical Fundament

Conforming the ontology of Mathematics is the non-ostensive nature of mathematical objects, from which the aspect of the epistemology of Mathematics is derived, which states that the appropriation of the concept in Mathematics is achieved through the representation of the object in different semiotic registers, Báez & Blanco [2], which leads to the need to use representations and specifically the registers of semiotic representation, as well as transfers between different registers of semiotic representation; the physical non-existence of mathematical objects is one of the reasons why the teaching of Mathematics presents difficulties worldwide; the aforementioned representations are the ways through which we can have access to mathematical objects and learn Mathematics Duval [5].

The analysis proposed in this paper is also based on the transition from procedural knowledge to conceptual knowledge, since as Vygotsky [16] stated, the concept in its finished form cannot be placed in the mind of the student, therefore, the Students have to work with algorithms and procedures through whose activity they will arrive at the formation of the concept. Procedural and conceptual knowledge will be understood here as expressed in Star & Stylianides [15]. Obviously, the student's activity on different aspects of the work object will be supported by representations and changes in semiotic representations.

Both the changes of representation registers and the transition from the process to the object are associated with the students achieving theoretical generalizations of what they have learned, taking into account that the theoretical generalization is the one that is carried out on the essential features of the phenomenon studied, while scientific concept is above the theoretical generalization of pre-concepts Karpov & Bransford [8].

3. Development

Definitions of Trigonometric Ratios

One of the problems faced by the teaching of Mathematics is the non-ostensive nature of mathematical objects, which makes it necessary to work with said objects through their semiotic materializations, so that the student appropriates a mathematical concept in general Duval [5]. You must work it in different representations, since each representation reveals different characteristics of the studied object; In the case of trigonometric functions, two alternatives are presented, they can be seen as ratios or as functions, they are seen as ratios when they are defined by the relationships between the sides and angles of the right triangle and as functions when they are defined by the axis systems coordinates, this last form is a generalization of the first and the student must see it that way, however, as stated in Sampaio & Batista [13], students tend to see the trigonometric ratios studied in high school as something different from the trigonometric functions studied at the pre-university level; In this sense, Maknun, Rosjanuardi, Jupri, [9] state that when students work with the angles referred to the triangle, this becomes an epistemological obstacle when they go on to study trigonometric functions, in the sense that propose Brousseau [3], according to Maknun, Rosjanuardi & Jupri [10], there was no analysis of students' understanding of trigonometry based on the epistemological obstacle.

However, in high school (when the study of trigonometry begins) you should start with the trigonometric ratios in order to make the content accessible to students, but from the beginning you should encourage students to obtain new relationships from given definitions, even with the teacher's guidance, for example, the fundamental identity $\sin^2(x) + \cos^2(x) = 1$, the student can obtain it relatively easily using the Pythagorean theorem, from which he can get $\tan^2(x)+1=\sec^2(x)$ and $1+\cot^2(x)=\csc^2(x)$ furthermore using the definitions the students should be able to get $\tan(x)=\dfrac{\sin(x)}{\cos(x)}$ y $\cot(x)=\dfrac{\cos(x)}{\sin(x)}$ as well as $\cos(x)\sec(x)=1$, $\sin(x)\csc(x) = 1$ and $\tan(x)\cot(x) = 1$. A task that does not by Classical is no longer important for students to train themselves in handling trigonometric ratios is to express each function in terms of the rest.

Following this idea, the values of the trigonometric ratios in notable angles should be ensured that students calculate them from an equilateral triangle with side one for the angles of 30° and 60° and an isosceles right triangle with leg one for the angle of 45° which would be the only thing they need to remember, as illustrated in the figures below:

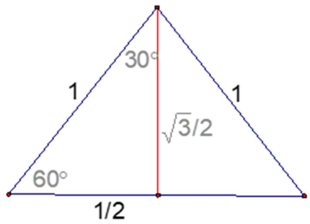

Figure 1. *Notable angles 30 and 60 degree.*

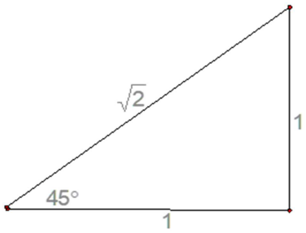

Figure 2. *Notable angles 45 degree.*

So that the use of notable values of the trigonometric ratios does not depend on memory or the use of tables for the work. It is clear that students currently use digital media to obtain the information they need, but despite the foregoing, the recommendations that are proposed are focused on students seeing Mathematics as it is, a medium and an object in itself and appreciating the resources available to them for mathematical work.

Another aspect that must be worked on at this level is the degree-radian relationship, but also looking for the reasoning of the students; It is usual to put students to work with radians without them having the slightest idea of what a radian is, so you should start by explaining what a radian is, that is: Angle measurement unit of the International System, of symbol rad, which is equivalent to a flat angle that, having its vertex in the center of a circle, corresponds to an arc of length equal to the radius of the circle.

Therefore the complete circle will have $\frac{2\pi r}{r}$ radians, that is $360° = \frac{2\pi r}{r}$, this is

$180° = \pi$ radians, and therefore a radian $= \frac{180^o}{\pi}$ this is $1° = \frac{\pi}{180}$ radians.

In this way the student has the tool to convert from one measure to the other as needed. If in the procedural activity the student does not work with the concept, she or he, will not be able to arrive at conceptual knowledge. Conventional trigonometry teaching has often been linked to procedural understanding based on memorizing rules and applying algorithms, instead of conceptual understanding based on establishing relationships between different mathematical concepts Nordlander [12].

Other results that students can obtain with the teacher's guidance are: the expression for calculating the area of a triangle, given two sides and the included angle, which is accessible to students by giving them the suggestion of constructing the height corresponding to one of the known sides; With practically the same idea, they can obtain the law of sines. They can also get the law of cosines with some algebraic work, which is necessary for students to internalize the systemic structure of Mathematics. Returning to what has been said about mathematical assistants (software for mathematical work), it is necessary to give students tasks that are not solved immediately by said assistants with very little use of the students' knowledge.

Trigonometry is conducive to many interesting application problems, but it is critical that students are involved in acquiring the mathematical tools they will need for these applications; The specialized bibliography abounds with examples of problems that are solved using trigonometric ratios, which of course should not be missing from a trigonometry course.

4. The Trigonometric Functions

The trigonometric functions must be introduced through the trigonometric circle in a Cartesian coordinate system, which allows students to determine the value of the trigonometric functions in the quadrant angles, as well as the positive or negative sign of the functions depending on the quadrant where be the terminal side of the angle without using memory; It also allows the foundation of their periodicity, which is an important characteristic of these functions, although when going to the trigonometric functions the students find it confusing that π, in addition to being worth 3.14 it also has the value of 180 degrees, although they learn to change the value of angles from degrees to radians and vice versa, they do so only at a procedural level without reaching the required concepts, so it is necessary to emphasize the concept of radian to students as stated in the previous section.

In order to generalize the trigonometric functions from the trigonometric circle, there are two aspects that must be made explicit to the students, first, why radius one can be taken to generalize the result of the values of the trigonometric functions and also why the values of the angles in the trigonometric circle allow to generalize the trigonometric functions to the coordinate system Maknun, Rosjanuardi & Jupri [9].

In order for students to objectively appreciate that the size of the radius of the circle does not change the value of the trigonometric functions, it is very convenient to use graphs in Geogebra or another mathematical assistant, as illustrated in the following graph:

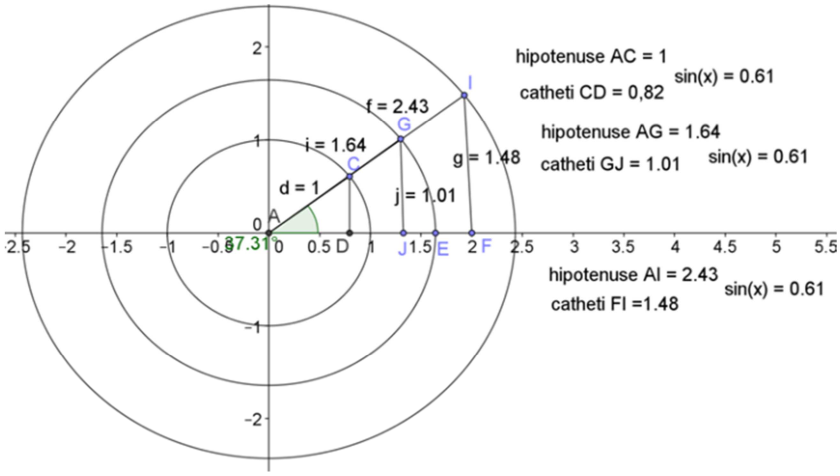

Figure 3. *Trigonometric circle.*

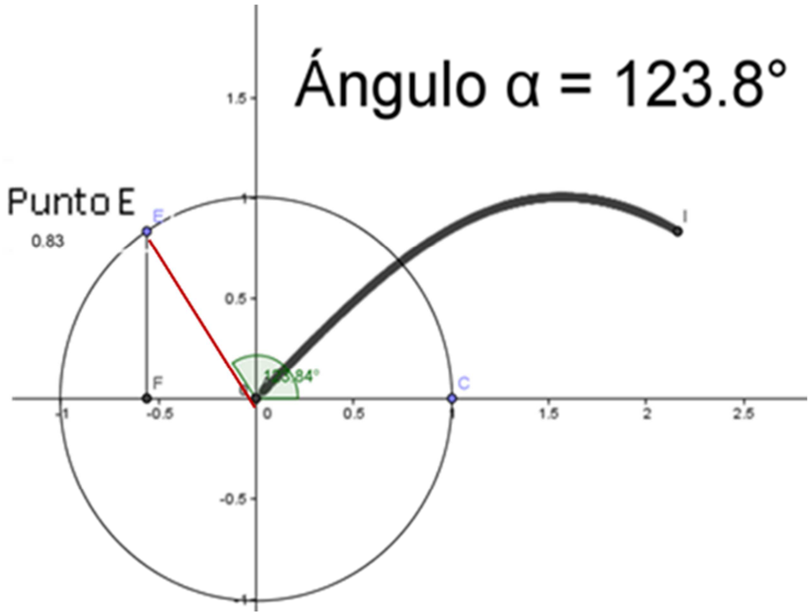

Figure 4. *Generation of sine function, angle up to 123 degree.*

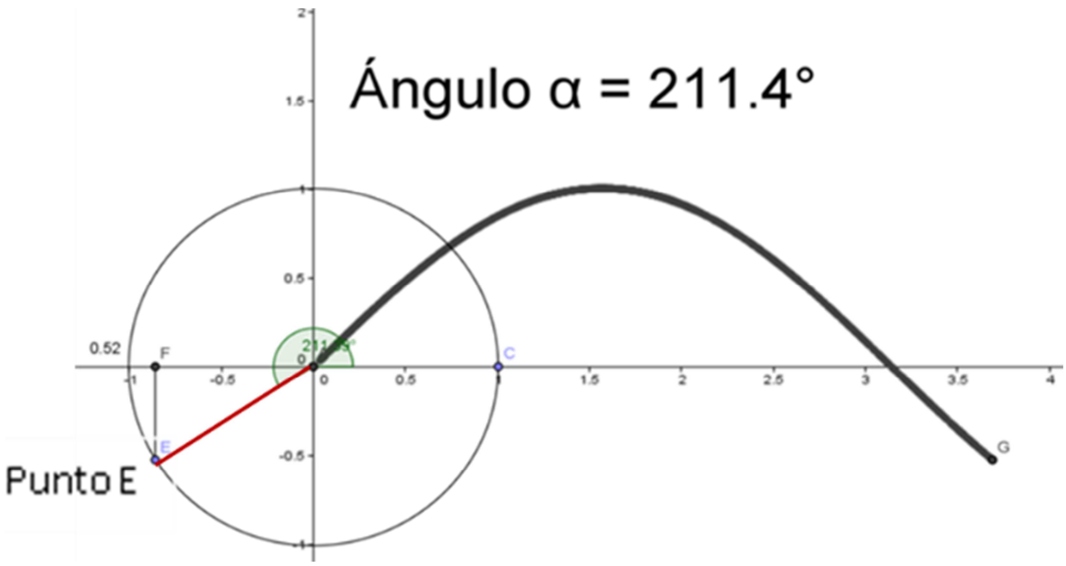

Figure 5. *Generation of sine function, angle up to 211 degree.*

Where it can be seen that the relationship between the radius of the circle and the leg of the triangle always give the same value of the sine of the angle; students can be oriented and in fact it is advisable that they do the same with the adjacent leg and with the angle in the different quadrants, where in addition to obtaining the same value for a given angle, regardless of the length of the radius of the circle, they will obtain the value of the function with the corresponding sign, of course, to obtain this result they have to take into account when the legs are positive or negative according to the coordinate system.

So that the students can appreciate how the graph of the trigonometric functions is generated from the expansion of the angle in the trigonometric circle, a dynamic graph can be used like the one shown (here it is shown in three steps, but having the software you can see how it generates the sine function by expanding the angle with center in the trigonometric circle). The effect is obtained by moving the point E which determines the amplitude of the angle and turning on the trace of the point.

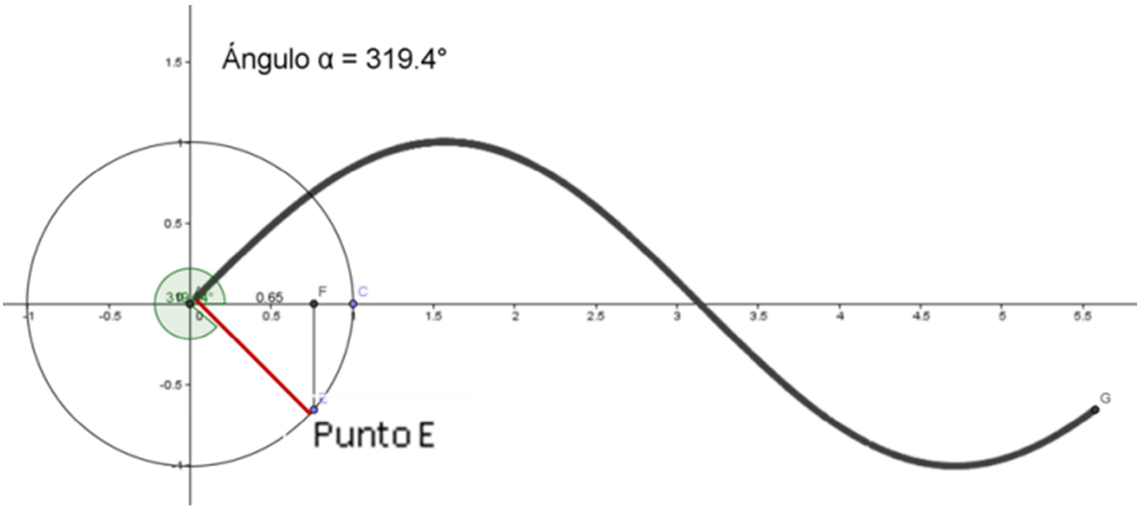

Figure 6. *Generation of sine function, angle up to 319 degree.*

Three stages of motion are shown here, but the best effect on students is achieved by using the software so that they can see, uninterruptedly, how the graph of the sine function is formed.

For students to understand many of the characteristics of trigonometric functions, it is useful, as stated in the previous illustrations, to rely on dynamic geometry software such as GeoGebra, with which the student can not only see the graphs but also to manipulate them to be able to appreciate different aspects that characterize these functions, for example, it is important that students see the difference between sin(nx) and n · sin(x), as illustrated in the following graphs:

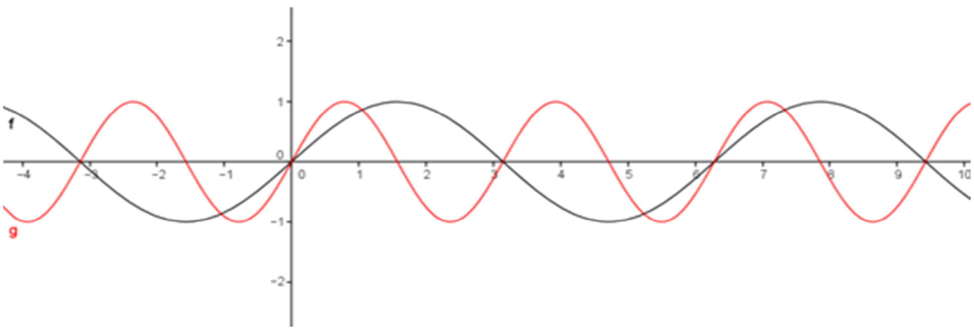

Figure 7. *Comparison of sin(nx).*

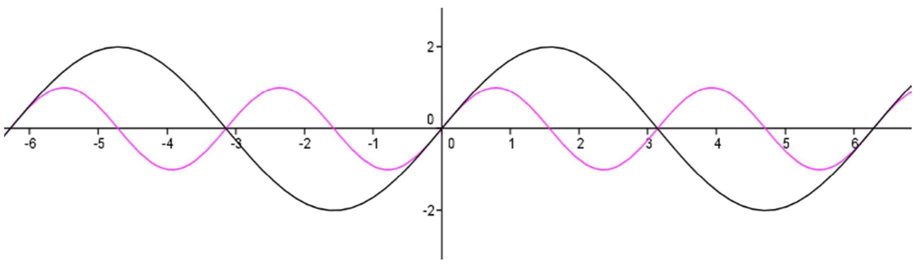

Figure 8. *Comparison of n · sin(x).*

In which sin(2x) and 2sin(x) are compared with the graph of sin(x) (in red) where the student can appreciate the effect of the coefficient, as part of the argument, or as a coefficient of the sine, done in the first case acts on the period and in the second on the expansion of the values of the ordinates, it also contributes a lot to the understanding of students who analyze graphs of functions such as y = a · sin(x+b) + c, y = a · cos(x+b) + c using a software such as GeoGebra, using sliders to assign values to the parameters, promotes a better interpretation of the characteristics of the graphs such as amplitude, intersections, periodicity, range by the students Mosese, & Ogbonnaya [11].

What has been explained in the previous paragraphs is based on the need for changes in semiotic registers for the correct interpretation of mathematical objects by students, as can be seen, the use of dynamic geometry software makes the procedural activity more efficient and also enjoyable for students.

5. Trigonometric Identities and Equations

Before beginning work with identities and equations, students should have worked with angle sum formulas, double angles, half angles, and transforming sums into products. In order for students to appreciate that Mathematics is built with mathematical tools and to make them participate in this construction, from the sin(x+y) and cos(x+y) formulas, students must obtain the remaining formulas necessary to work with the identities and equations.

5.1. Trigonometric Equations

The first obstacle that students face when working with these equations is considering the solutions only in the interval between 0 and 90 degrees, or at most between 0 and 360 degrees, so that they take into account all possible solutions to a graph like the one below, will allow them to appreciate the need to take into account the range of the trigonometric functions:

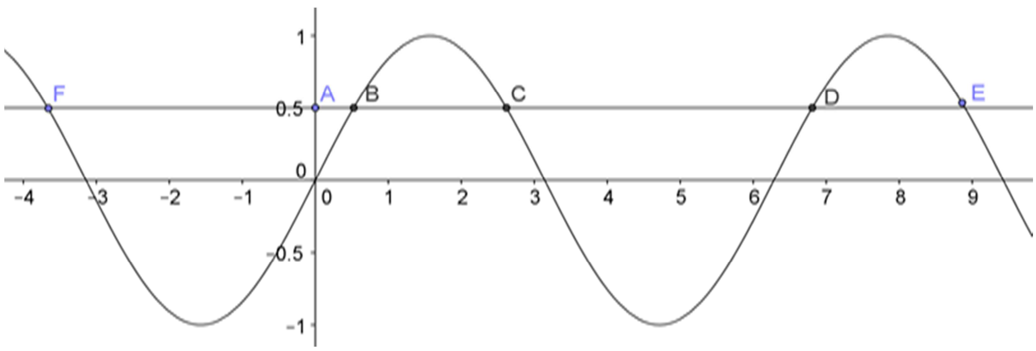

Figure 9. *Same value of the function in different angles.*

For students to begin to solve trigonometric equations, there are two necessary preliminary orientations, the first is the concept of solving an equation, an orientation that must be focused on in a general way, that is, the solution of an equation (whatever it may be) is the value or values that substituted in the equation convert it into an identity; The second is to guide that the resolution process is based on expressing the equation in a single trigonometric function, for example: $sin^2(x)+\cos(x)=1$ which is solved by substituting the $sin^2(x)$ for $1-\cos^2(x)$ with what you get: $1-\cos^2(x)+\cos(x)=1$ simplifying and factoring $\cos(x)(1-\cos(x)=0$, this is: cos(x) = 0 y cos(x) =1; now it must be taken into account that the solution is associated to the periodicity of the cosine, therefore of cos(x) = 0 solutions are derived: x= $\frac{\pi}{2}\pm n\pi$ for integer values of n and from cos(x) = 1 are derived the solutions x = $\pm 2n\pi$.

In solving trigonometric equations, a resource of general application in solving mathematical problems is also useful, this is the change of variable, as is the case in the following example: $\tan^2(x)-2\tan(x)+1=0$, making the change of variable tan(x) = u and substituting into the equation, we get: $u^2-2u+1=0$, namely $(u-1)^2=0$ which has the double root u = 1 returning to the original variable tan(x) = 1 from which the solutions x = $\frac{\pi}{4}\pm n\pi$ are derived. In the bibliography, there are many materials with a great variety of equations that have been solved and to be solved, from which teachers can choose the appropriate equations to assign the tasks to their students, taking into account that the chosen examples do not repeat the same procedures and leave alternatives that they do not exercise.

Although it is necessary to emphasize the usefulness of formulating equations that imply some algebraic work, so that students appreciate the systemic structure of Mathematics, as in the following example: $\sqrt{1-sen(x)}\sqrt{1+sen(x)}$ = 1, from where is it obtained $\sqrt{\cos^2(x)}=1$ namely $|\cos(x)|=1$ where it may be necessary to remind students that $\sqrt{a^2}=|a|$ so the solution is x = $n\pi$.

5.2. Trigonometric Identities

As with the trigonometric equations, the bibliography is abundant with examples of trigonometric identities, although there are two precisions that should be oriented to the students, this is how it is tried to prove that the equality between the two members of the same is verified, you cannot pass terms from one member to the other, you have to work on one member until you reach the other or work on both until you achieve equality, it is also possible to pass the two members to the same side of the equality and prove that the subtraction is equal to zero, because if A – B = 0 then A = B; Given that these pathways are widely covered in the bibliography, they will not be detailed here, however we will show a pathway that is not frequently used but that is useful both to prove the identity itself, so that students do not disassociate themselves from the trigonometric circle, maintain or develop algebraic technical skills and appreciate the systemic structure of mathematics, Blanco [2].

The aforementioned procedure is illustrated below: Prove the following identity $\cot(\alpha)+\tan(\alpha)=\sec(\alpha)\cdot\csc(\alpha)$, Instead of making trigonometric transformations, each of the functions is expressed by its corresponding values on the trigonometric circle: $\cos(\alpha)=x$, $\sin(\alpha)=y$ as $\cot(\alpha)=\dfrac{\cos(\alpha)}{\sin(\alpha)}$ is obtained $\cot(\alpha)=\dfrac{x}{y}$ in the same way $\tan(\alpha)=\dfrac{y}{x}$, then substituting on the right hand side of the identity $\dfrac{x}{y}+\dfrac{y}{x}=\dfrac{xx}{xy}+\dfrac{yy}{xy}=\dfrac{x^2+y^2}{xy}$ taking into account the relationship of the variables in the trigonometric circle $\dfrac{1}{xy}=\dfrac{1}{x}\dfrac{1}{y}$ using the relationships on the trigonometric circle $\sec(\alpha)\cdot\csc(\alpha)$, identity being proven, Goel & Elstak [6].

The method is even more efficient when powers appear as in the following example: Prove the identity $\dfrac{\cot^2(\alpha)}{1+\csc(\alpha)}=\dfrac{1-\sin(\alpha)}{\sin(\alpha)}$, proceeding as in the previous example $\dfrac{\frac{x^2}{y^2}}{1+\frac{1}{y}}$ multiplying numerator and denominator by y^2 we have $\dfrac{\frac{x^2}{y^2}y^2}{\left(1+\frac{1}{y}\right)y^2}=\dfrac{x^2}{y^2+y}$, as $x^2+y^2=1$ it is achieved $x^2=1-y^2$ then substituting and factoring we finally have $\dfrac{1-y}{y}=\dfrac{1-\sin(\alpha)}{\sin(\alpha)}$ with which the identity is proven.

Here it is appropriate to make a consideration to achieve precision in writing, when working with powers of trigonometric functions, the exponent must be written over the function, as stated in the example. $cot^2(x)$ and not as it appears in some texts $cot\alpha^2$ because it is natural to think that it is the angle that is raised to the square.

The procedure is also applicable when functions evaluated in more than one argument appear in the identity, an example is given: Prove the identity: $\dfrac{\sin(A)\cos(B)+\cos(A)\sin(B)}{\cos(A)\cos(B)-\sin(A)\sin(B)}=\dfrac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$ proceed as in the previous examples, it is only necessary to distinguish the representation for each argument, that is: $\dfrac{x_1y_2+y_1x_2}{x_1x_2-y_1y_2}$ now dividing numerator and denominator by x_1x_2 we get: $\dfrac{x_1y_2+y_1x_2}{x_1x_2-y_1y_2}=\dfrac{\frac{y_2}{x_2}+\frac{y_1}{x_1}}{1-\frac{y_1y_2}{x_1x_2}}=\dfrac{\tan(A)+\tan(B)}{1-\tan(A)\cdot\tan(B)}$ and the identity is proven:

The procedure explained here has the following advantages: 1) it is an alternative for student work; 2) helps students develop or maintain algebraic technicality; 3) shows the usefulness of using the trigonometric circle turning it into a work tool.

6. Verification of the Effectiveness of the Proposed Guidelines

In the development of the master's degree: "Specialty in teaching basic and upper secondary Mathematics" developed at the APEC university (Action for Education and Culture), the teachers participating in the master's degree were oriented to teach trigonometry, emphasizing the aspects of this branch of Mathematics that affect the difficulties that students present in their learning, indicated in the present work. Subsequently, the teachers who teach the subject were interviewed about the application and result of the aforementioned guidelines, all the interviewees stated that they had better results in their students' knowledge of trigonometry when teaching the subject applying what they learned in the master's degree in this regard; which allows us to ensure how useful these guidelines are in learning trigonometry.

7. Conclusions

As can be seen, this article is not a trigonometry course, the classic examples and exercises are left to the teachers' choice, using the existing bibliography on the subject.

The aspects dealt with based on theoretical foundations such as the transfer of semiotic registers, supported by mathematics software, the transition from the process to the object and ontological characteristics of Mathematics such as its interlinked systemic structure, which enables students to make theoretical generalizations, justify the advantages of teaching trigonometry content following the guidelines provided, which allows us to propose the achievement of the proposed objective: to provide a set of didactic guidelines, with its theoretical foundations, that help teachers in the direction of the teaching-learning process of the trigonometry.

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